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Steady free convection above a point heat source and a horizontal line heat source in a vertical magnetic field

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An electrically conducting fluid is contained above a horizontal plane. A uniform vertical magnetic field is applied externally and the plane is maintained at a uniform temperature except for a point or a line heat source. Density variations are ignored except where they give rise to buoyancy forces.

(i) *The point heat source.* Non-linear effects are small sufficiently far from the source. The resulting buoyancy forces interact with the magnetic forces to maintain a radial inflow towards the heat source. This fluid then escapes vertically as a jet, its structure now depending on the additional influence of viscosity. The perturbations of the temperature distribution and the magnetic field due to the motion are obtained. Finally, the effects of these perturbations back on to the fluid velocity are considered. The most striking features of the perturbations are (*a*) the action of the jet as a line source of heat for the fluid in the outer regions, (*b*) the large (compared to other perturbations) eddy in the jet.

(ii) *The line heat source.* The temperature distribution and magnetic field are weakly perturbed only if the thermal and electrical conductivities are sufficiently small. Similar results are obtained, as in (i) above, provided ϵ (a dimensionless number characterising the strength of thermal convection: see (1.32), (3.24)) is less than $\frac{1}{4}$. However, even for small ϵ, the effects of thermal convection cannot be ignored. Hence, superimposed on the jet is an eddy (driven by buoyancy forces) whose flux of fluid increases indefinitely with its height above the plane. When $\epsilon > \frac{1}{4}$, the results suggest that numerous eddies will be formed.

1. Introduction

Thermal convection in a strong magnetic field has much astrophysical interest, since in a star thermal convection may be considerably influenced by the action of magnetic forces. Although stability problems of the type considered by Chandrasekhar (1961) have been studied extensively, the problem of steady convection in a uniform vertical magnetic field does not appear to have been investigated. In this paper attention is restricted to the following two problems: An electrically conducting fluid is contained above an infinite horizontal plane. A uniform vertical magnetic field is applied externally, and the plane is maintained at a uniform temperature except for a point or line heat source. The density of the fluid is assumed constant except where slight density variations caused by the heating give rise to buoyancy forces (the Boussinesq approximation). The steady motion resulting from the differential heating is examined.

Free convection in the absence of magnetic forces has been studied by many authors, but of particular interest here is the work of Fujii (1963). He considered free convection above both a line and a point source of heat, and showed that the motion consists of a buoyant plume which entrains fluid horizontally from infinity. The thermal boundary-layer approximations were made in the equations governing the motion of the plume, and hence similarity solutions were obtained. Since motion across the magnetic field lines is inhibited, this model must be modified considerably in the presence of a vertical magnetic field.

The problem of a jet in an aligned magnetic field has certain similarities with thermal convection above a heat source. Toomre (1967) considered a two-dimensional jet, and showed that far downstream inertia is negligible. In this region a similarity solution (also proposed independently by Jungclaus 1965) was obtained, in which a balance of magnetic and viscous forces was maintained. Hoult (1965) had attempted to describe the motion of a round jet (for the particular case of large Reynolds number and small magnetic Reynolds number) near the axis of symmetry by Schlichting's (1955) classical non-magnetic solution. Since this demands the entrainment of fluid, a solution was sought outside the jet that corresponded to a uniform line sink along the axis. Unfortunately, the non-linear ordinary differential equation, which he solved numerically to determine the motion, has no solution satisfying the required boundary conditions. The approach adopted by Hoult, though attractive, is certain to encounter the same difficulties in the case of the buoyant plume. Essentially, entrainment of such a large quantity of fluid cannot be maintained without convecting the magnetic field lines.

(i) *The case of the point heat source.* Cylindrical polar co-ordinates (r, ϕ, z) are taken with $\hat{\mathbf{z}}$ (the unit vector in the z direction) vertical and with the source of heat at the origin $(0,0,0)$. It is supposed that the space $z < 0$ is occupied by a solid and that a uniform magnetic field $B_0\hat{\mathbf{z}}$ is maintained as $z \to \pm\infty$. For steady flow, Maxwell's equations reduce to

$$\nabla . \mathbf{B} = 0, \quad \mathbf{j} = \mu^{-1}\nabla \wedge \mathbf{B}, \quad \mathbf{E} = -\nabla\Phi, \tag{1.1}$$

while Ohm's law states that

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \tag{1.2}$$

where μ is the magnetic permeability, σ is the electrical conductivity, $\mathbf{E}$ is the electric field, Φ is the electric potential, $\mathbf{j}$ is the electric current, $\mathbf{u} = (u_r, 0, u_z)$ is the fluid velocity and $\mathbf{B} = (B_r, 0, B_z)$ is the magnetic field. Since (1.1), (1.2) and the axisymmetry lead to

$$\begin{aligned} \nabla^2\Phi &= \nabla . (\mathbf{u} \wedge \mathbf{B}), \\ &= 0, \end{aligned} \tag{1.3}$$

it follows that the electric field is uneffected by the fluid velocity. Hence provided no external electric field is applied

$$\mathbf{E} = 0. \tag{1.4}$$

Further, since $\mathbf{u} = 0$ for $z < 0$, (1.2) shows that

$$\mathbf{j} = 0 \quad (z < 0). \tag{1.5}$$

Making the Boussinesq approximation we have the equation of motion,

$$\rho(\mathbf{u}.\nabla)\,\mathbf{u} = -\nabla p + \rho g \alpha T\hat{\mathbf{z}} + \mathbf{j}\wedge\mathbf{B} + \rho\nu\nabla^2\mathbf{u}, \tag{1.6}$$

where ρ is the density, ν is the kinematic viscosity, α is the coefficient of expansion, g is the acceleration due to gravity, T is the excess temperature and $(p-\rho g z)$ is the pressure. The remaining equations governing the motion are the continuity equation,

$$\nabla.\mathbf{u} = 0, \tag{1.7}$$

and the heat conduction equation,

$$(\mathbf{u}.\nabla)\,T = K\nabla^2 T, \tag{1.8}$$

where K is the thermal conductivity.

The boundary conditions on $z = 0$ are

$$T(r,\phi,0) = 0 \quad (r>0), \quad Q = \int_0^\infty rT(r,0,0)\,dr, \tag{1.9}$$

where $2\pi Q$ is a measure of the total input of heat,

$$u_z = u_r = 0, \tag{1.10}$$

$$\text{and the continuity of both } B_r \text{ and } B_z \text{ across } z = 0 \tag{1.11}$$

(see (1.27) together with the footnote), while the boundary conditions as $|\mathbf{r}| \to \infty$ are

$$\mathbf{u}\to 0, \quad \mathbf{B}\to B_0\hat{\mathbf{z}}, \quad T\to 0. \tag{1.12}$$

From (1.1), (1.2) and (1.6) it is evident that the magnetic and viscous forces determine a length scale,

$$l = \{\rho\nu/\sigma B_0^2\}^{\frac{1}{2}}. \tag{1.13}$$

This is the familiar Hartmann length scale, over which these forces can be in equilibrium. Since we are primarily concerned with the interaction of buoyancy and magnetic forces, the following dimensionless variables are introduced:

$$\left.\begin{aligned} \mathbf{r}' &= \mathbf{r}/l, \quad \mathbf{u}' = (\nu/g\alpha Q)\,\mathbf{u}, \quad \mathbf{b} = \mathbf{B}/B_0, \\ \mathbf{j}' &= (\nu/g\alpha Q)\,(\sigma B_0)^{-1}\mathbf{j}, \quad \theta = (l^2/Q)\,T, \quad p' = (l/\rho g\alpha Q)\,p, \end{aligned}\right\} \tag{1.14}$$

together with the only three independent dimensionless numbers,

$$P = \nu/K, \quad \beta = lg\alpha Q/\nu K, \quad \gamma = lg\alpha Q/\nu(\sigma\mu)^{-1}, \tag{1.15}$$

where P is the Prandtl number and β is the Rayleigh number based on the Hartmann length scale. The governing equations now become (after dropping the primes)

$$\nabla.\mathbf{u} = 0, \quad \nabla.\mathbf{b} = 0, \tag{1.16}$$

$$\beta P^{-1}(\mathbf{u}.\nabla)\,\mathbf{u} = -\nabla p + \theta\hat{\mathbf{z}} + (\mathbf{u}\wedge\mathbf{b})\wedge\mathbf{b} + \nabla^2\mathbf{u}, \tag{1.17}$$

$$\gamma\mathbf{u}\wedge\mathbf{b} = \nabla\wedge\mathbf{b}, \tag{1.18}$$

$$\beta(\mathbf{u}.\nabla)\,\theta = \nabla^2\theta. \tag{1.19}$$

The stream function ψ and the magnetic vector potential $(0, \chi/r, 0)$ are introduced so that the velocity and magnetic field are given by

$$\mathbf{u} = \left(-\frac{1}{r}\frac{\partial\psi}{\partial z}, 0, \frac{1}{r}\frac{\partial\psi}{\partial r}\right), \quad \mathbf{b} = \left(-\frac{1}{r}\frac{\partial\chi}{\partial z}, 0, \frac{1}{r}\frac{\partial\chi}{\partial r}\right). \tag{1.20}$$

The curl of (1.17) is taken, and after some manipulation (1.17)–(1.19) become

$$\beta P^{-1} r\left\{-\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right\}\left(\frac{D^2\psi}{r^2}\right) = r\frac{\partial\theta}{\partial r} - r^2(\mathbf{b}.\nabla)\left\{\frac{1}{r^2}(\mathbf{b}.\nabla)\psi\right\} + D^2(D^2\psi), \quad (1.21)$$

$$\left.\begin{aligned}\frac{\gamma}{r}\left\{-\frac{\partial\psi}{\partial z}\frac{\partial\chi}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial\chi}{\partial z}\right\} &= D^2\chi \quad (z>0),\\ 0 &= D^2\chi \quad (z<0),\end{aligned}\right\} \quad (1.22)$$

$$\frac{\beta}{r}\left\{-\frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial r}+\frac{\partial\psi}{\partial r}\frac{\partial\theta}{\partial z}\right\} = \nabla^2\theta, \quad (1.23)$$

where

$$D^2 = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}, \quad (1.24)$$

and

$$(\mathbf{b}.\nabla) = -\frac{1}{r}\frac{\partial\chi}{\partial z}\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial\chi}{\partial r}\frac{\partial}{\partial z}. \quad (1.25)$$

In terms of the new variables the boundary conditions become

$$\frac{\partial\theta}{\partial r} = \frac{\partial u_z}{\partial r} = 0 \quad \text{on} \quad r = 0,\dagger \quad (1.26)$$

$$\left.\begin{aligned}\theta = 0 \quad (r>0), \quad \int_0^\infty r\theta\,dr &= 1,\\ u_r = u_z &= 0,\end{aligned}\right\} \quad \text{on} \quad z=0, \quad (1.27)$$

continuity of **b**‡

$$\mathbf{u}\to 0, \quad \mathbf{b}\to\hat{\mathbf{z}}, \quad \theta\to 0 \quad \text{as} \quad |\mathbf{r}|\to\infty. \quad (1.28)$$

(ii) *The case of the line heat source.* The two-dimensional problem is very similar. Co-ordinates (x, y, z) are taken with $\hat{\mathbf{y}}$ (the unit vector in the y direction) vertical and with the line source of heat at $(0, 0, z)$. Attention is restricted to the case where the solid is a perfect conductor (see § 3.3). Thus the only boundary condition that is applied on the magnetic field at $z = 0$ is b_y = constant (= strength of the uniform vertical magnetic field in the solid).

Most of the equations in the cylindrical geometry carry over to the two-dimensional case with certain slight alterations. In particular, the boundary conditions (1.9) and (1.11) are replaced by

$$T(x, 0) = Q\delta(x), \quad (1.29)$$

and

$$B_y(x, 0) = \text{constant}. \quad (1.30)$$

The equations governing the motion are made dimensionless by the substitutions (the dimensions of Q are different)

$$\left.\begin{aligned}\mathbf{x}' = \mathbf{x}/l, \quad \mathbf{u}' = l^{-1}(\nu/g\alpha Q)\,\mathbf{u}, \quad \mathbf{b} = \mathbf{B}/B_0,\\ \mathbf{j}' = (\sigma B_0 l)^{-1}(\nu/g\alpha Q)\,\mathbf{j}, \quad \theta = (l/Q)T, \quad p' = (\rho g\alpha Q)^{-1}p,\end{aligned}\right\} \quad (1.31)$$

† These are just symmetry conditions.

‡ If the solid is a perfect conductor, the magnetic field in the solid is $\mathbf{b} = \hat{\mathbf{z}}$ and this boundary condition is replaced by $b_z = 1$ on $z = 0$. The tangential component of the magnetic field may be discontinuous, resulting in a surface current sheet on the conductor.

and hence the dimensionless numbers (1.15) are replaced by

$$\epsilon_0 = l^2 g\alpha Q/\nu\kappa, \quad \alpha_0 = l^2 g\alpha Q/\nu(\sigma\mu)^{-1}. \tag{1.32}$$

Defining the velocity and magnetic field as

$$\mathbf{u} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right), \quad \mathbf{b} = \left(\frac{\partial\chi}{\partial y}, -\frac{\partial\chi}{\partial x}, 0\right), \tag{1.33}$$

the governing equations (after some manipulation) reduce to

$$-\epsilon_0 P^{-1}(\mathbf{u}.\nabla)\nabla^2\psi = \frac{\partial\theta}{\partial x} + (\mathbf{b}.\nabla)\{(\mathbf{b}.\nabla)\psi\} - \nabla^2(\nabla^2\psi), \tag{1.34}$$

$$\alpha_0(\mathbf{u}.\nabla)\chi = \nabla^2\chi, \tag{1.35}$$

$$\epsilon_0(\mathbf{u}.\nabla)\theta = \nabla^2\theta, \tag{1.36}$$

where

$$(\mathbf{u}.\nabla) = \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}, \tag{1.37}$$

$$(\mathbf{b}.\nabla) = \frac{\partial\chi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\chi}{\partial x}\frac{\partial}{\partial y}, \tag{1.38}$$

together with the boundary conditions

$$\frac{\partial\theta}{\partial x} = \frac{\partial u_y}{\partial x} = 0 \quad \text{on} \quad x = 0, \tag{1.39}$$

$$\left.\begin{aligned} \theta &= \delta(x), \\ u_x &= u_y = 0, \\ b_y &= \text{constant} \end{aligned}\right\} \quad \text{on} \quad y = 0, \tag{1.40}$$

and

$$\left.\begin{aligned} \mathbf{u} \to 0 \quad \theta \to 0 \quad &\text{as} \quad |\mathbf{x}| \to \infty, \\ \mathbf{b} \to \hat{\mathbf{y}} \quad &\text{as} \quad y \to \infty. \end{aligned}\right\} \tag{1.41}$$

In the following sections solutions of the problems posed by (1.20)–(1.28) and (1.33)–(1.41) are presented, which are valid well clear of the Hartmann layer, i.e. $|z| \gg 1$ (or $y \gg 1$).

2. The point source of heat

2.1. *The first approximation*

If a plume of the type described by Fujii (1963) exists, outward diffusion of heat from the plume must be overcome by the inflow of heat by convection. Moreover, since the temperature is uniform outside the plume, there can be no buoyancy forces acting in this region. Hence, entrainment of fluid into the plume must be maintained by considerable pressure forces in order to overcome the action of the magnetic field. It would seem unlikely, therefore, that this model could represent the resulting flow, and we should expect that thermal diffusion is at least comparable to convection in the main body of the fluid.

Assume, for the moment, that the magnetic field is unperturbed. Then, assuming that the dimensionless numbers P, β, γ are of order 1, the nature of the boundary conditions suggests that the motion may be described by similarity solutions of the form,

$$\psi = z^s\Psi_0(\eta), \quad \theta = z^t\Theta_0(\eta), \tag{2.1}$$

for $R \gg 1$, where
$$\eta = r/z \quad \text{and} \quad R = (r^2+z^2)^{\frac{1}{2}}. \tag{2.2}$$

Further, considering the inhibiting effect of the magnetic field on the flow, there appears a strong possibility that the effects of thermal convection will be negligible. Thus, since the boundary condition (1.27) implies

$$\lim_{z\to 0}\int_0^\infty r\theta\, dr = 1, \tag{2.3}$$

it follows that
$$\theta = z^{-2}\Theta_0(\eta). \tag{2.4}$$

Now with this value of θ it is clear that the magnetic and buoyancy forces are comparable in the equation of motion, provided

$$\psi = \Psi_0(\eta). \tag{2.5}$$

Substituting the above values of ψ and θ into the governing equations (1.21) and (1.23), and retaining only the highest powers of z, we have

$$(\eta\Psi_0)'' - \Theta_0' = 0, \tag{2.6}$$

$$(1+\eta^2)\,\Theta_0'' + (\eta^{-1}+6\eta)\,\Theta_0' + 6\Theta_0 = 0. \tag{2.7}$$

The solution of (2.7), satisfying the boundary condition (2.3), is

$$\Theta_0 = (1+\eta^2)^{-\frac{3}{2}}. \tag{2.8}$$

Neglecting the no-slip condition on $z = 0$, but retaining the condition $\psi = 0$, (2.6) integrates to give
$$\Psi_0 = (1+\eta^2)^{-\frac{1}{2}}. \tag{2.9}$$

It is evident that the boundary conditions on ψ at $r = 0$ are not satisfied, so that a boundary-layer solution must describe the motion near $r = 0$. Suppose that the horizontal length scale is H, and that the vertical length scale is L ($\gg H$). Then the ratio of viscous to inertia forces in (1.21) is of order (LP/β) (since $\Psi_0(0) = 1$). Thus, inertia forces will be negligible, and a balance of viscous, magnetic and buoyancy forces will be maintained. Since the ratio of magnetic to viscous forces is of order (H^4/L^2), it seems reasonable to look for similarity solutions of the form,
$$\psi = \psi_0(\xi), \quad \theta = z^{-2}\theta_0(\xi), \tag{2.10}$$

where $\xi = r/2z^{\frac{1}{2}}$. Substituting into (1.21), (1.23), and as before retaining only the highest powers of z, we are left with

$$\left[-1-\frac{\xi}{2}\frac{d}{d\xi}+\frac{\xi}{4}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)\right]\left[-\frac{\xi}{2}\frac{d}{d\xi}-\frac{\xi}{4}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)\right]\psi_0 = \xi\theta_0', \tag{2.11}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta_0}{d\xi}\right) = 0. \tag{2.12}$$

Applying the boundary conditions (1.26) and $\psi_0(\infty) = 1$, these have the solutions,

$$\psi_0 = 1-e^{-\xi^2}, \quad \theta_0 = 1, \tag{2.13}$$

where ψ_0 is axisymmetric analogue of the solution proposed by Jungclaus (1965) for the two-dimensional jet. Moreover, defining

$$\chi = \begin{cases} z^2\,\Phi_0(\eta) & \text{outside the jet,} \\ z\,\phi_0(\xi) & \text{inside the jet,} \end{cases} \tag{2.14}$$

it is apparent that, to a first approximation,

$$D^2\chi = 0. \tag{2.15}$$

Hence the values of θ and ψ given by (2.8), (2.9) and (2.13) provide a self-consistent description of the motion outside the Hartmann layer ($z \gg 1$). In

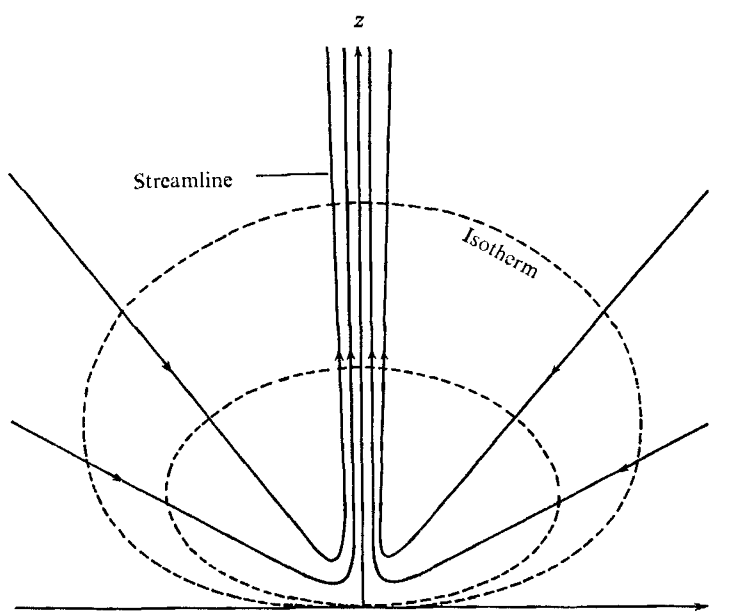

FIGURE 1. The streamlines and isotherms above the point heat source obtained from the first approximation.

particular, the non-linear effects are negligible, so that the temperature distribution ($\theta = z/R^3$), and the magnetic field ($\chi = \frac{1}{2}r^2$), are independent of the fluid velocity. Outside the jet, the vertical component of the momentum equation (1.6),

$$0 = -\frac{\partial p}{\partial z} + \theta, \tag{2.16}$$

determines the pressure distribution,

$$p = -1/R. \tag{2.17}$$

Hence, the radial balance of magnetic and pressure forces

$$0 = -\frac{\partial p}{\partial r} - u_r, \tag{2.18}$$

together with continuity, determines a radial inflow of fluid with velocity

$$u_R = -1/R^2. \tag{2.19}$$

The fluid is then ejected along the axis $r = 0$, by the action of the pressure force,

$$p = -\frac{1}{z}+\frac{1}{2z}e^{-\xi^2}. \tag{2.20}$$

The first term produces a force which exactly balances the buoyancy force, while the second term balances viscous forces and magnetic forces in the vertical and radial directions respectively.

Finally, the total vertical flux of heat,

$$F = \int_0^\infty \left(\beta u_z \theta - \frac{\partial \theta}{\partial z}\right) r\,dr, \tag{2.21}$$

is to lowest order in z given by

$$F = -\int_0^\infty \frac{\partial}{\partial z}\left(\frac{z}{R^3}\right) r\,dr = 0, \tag{2.22}$$

i.e. all the heat that is put in at the source is extracted throughout the remainder of the plane $z = 0$.

2.2. *The second approximation*

The analysis of the previous section suggests that a systematic approach may be adopted to obtain a higher order approximation. We divide the space $z > 0$ into the following regions (see figure 2)

$$\left.\begin{array}{lll} \mathrm{I} & z \gg 1, & r \gg z^{\frac{1}{2}}, \\ \mathrm{II} & z \gg 1, & r \ll z, \\ \mathrm{III} & r \gg 1, & z = O(1), \\ \mathrm{IV} & r = O(1), & z = O(1), \end{array}\right\} \tag{2.23}$$

and define the overlap of regions I and II as region V. Provided we neglect the no-slip condition on $z = 0$, we may ignore the Hartmann layer in region III. Moreover, it is reasonable to assume that the flow for sufficiently large R does not depend on the detailed nature of the flow in region IV. Hence, boundary conditions to the flow in regions I and II are prescribed only on $z = 0$ and $r = 0$. We now assume that the solutions may be written in the form

$$\psi = \Psi_0(\eta)+\zeta_{01}(z^{-1})\,\Psi_1(\eta)+\zeta_{02}(z^{-1})\,\Psi_2(\eta)+\ldots, \tag{2.24}$$

in region I, and

$$\psi = \psi_0(\xi)+\zeta_{11}(z^{-1})\,\psi_1(\xi)+\zeta_{12}(z^{-1})\,\psi_2(\xi)+\ldots, \tag{2.25}$$

in region II, where $\zeta_{0n} \gg \zeta_{0n+1}$ etc. Since both these equations must be a valid description of the flow in region V, they must be approximately equivalent when $\xi \gg 1$, $\eta \ll 1$. The method of matching adopted is that of Van Dyke (1964), where ξ, η are regarded as inner and outer variables respectively, and z^{-1} is regarded as a small parameter.

Making the above assumptions, we proceed to obtain first corrections to the temperature distribution, the magnetic field and the velocity profile. Since the corrections are found to be uniformly small for $|z| \gg 1$, these results give weight to the lowest order solutions. Further, since this method (used implicitly in

§ 2.1) is not well established, confidence is gained by forming a consistent second approximation.

(i) *The temperature distribution.* We suppose that

$$\theta = z^{-2}\,\Theta_0(\eta)+\beta z^{-3}\,\Theta_1(\eta)+\ldots, \tag{2.26}$$

in region I, and

$$\theta = z^{-2}+\beta A z^{-3}\ln z+\beta z^{-3}\theta_1(\xi)+z^{-3}g(\xi)+\ldots, \tag{2.27}$$

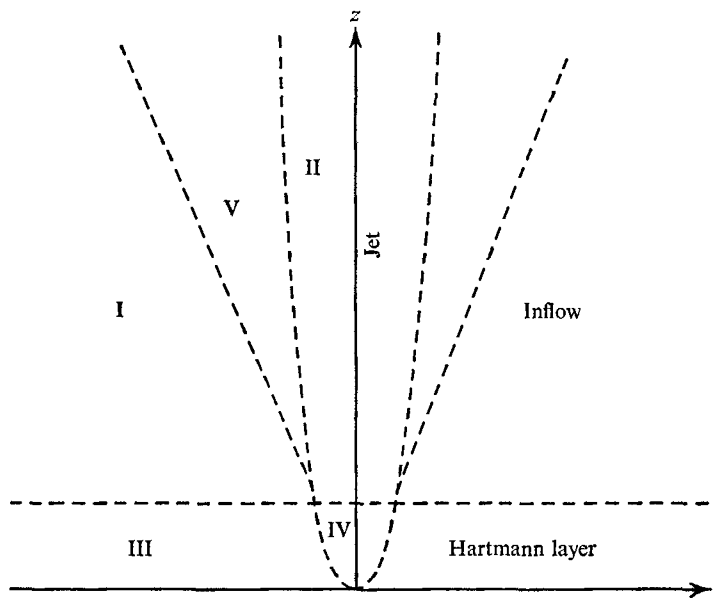

FIGURE 2. The various regions above the point heat source.

in region II. The necessity for the term $\beta A z^{-3}\ln z$ (A being a constant) becomes apparent later, while the term $z^{-3}g(\xi)$ is determined by the vertical diffusion of the lowest order temperature. Now direct substitution into (1.23) leads to

$$(1+\eta^2)\,\Theta_1''+(\eta^{-1}+8\eta)\,\Theta_1'+12\Theta_1 = -2\eta^{-1}\Psi_0'\,\Theta_0, \tag{2.28}$$

$$\tfrac{1}{4}\xi^{-1}(\xi g')' = -z^4\frac{d^2}{dz^2}\left(\frac{1}{z^2}\right) = -6, \tag{2.29}$$

$$(\xi\theta_1')' = -2\psi_0', \tag{2.30}$$

together with the boundary conditions

$$\theta_1' = g' = 0 \quad \text{at} \quad \xi = 0, \tag{2.31}$$

$$g \to -6\xi^2 \quad \text{as} \quad \xi \to \infty,\dagger \tag{2.32}$$

$$\Theta_1 = O(\eta^{-4}) \quad \text{as} \quad \eta \to \infty. \tag{2.33}$$

Integration of (2.28) is made easier by the substitution,

$$u^2 = 1+\eta^2, \tag{2.34}$$

† By matching with outer lowest order temperature.

and leads to the differential equation,

$$(u^2-1)\frac{d^2\Theta_1}{du^2}+(8u-6u^{-1})\frac{d\Theta_1}{du}+12\Theta_1 = -\frac{2}{u}\frac{d\Psi_0}{du}\Theta_0, \tag{2.35}$$

with the general solution,

$$\Theta_1 = \frac{1}{2u^4}+B\frac{3-u^2}{u^5}+C\left\{\frac{6}{u^4}+\frac{3-u^2}{u^5}\ln\left(\frac{u-1}{u+1}\right)\right\}. \tag{2.36}$$

The boundary condition (2.33) implies that

$$B = 0, \tag{2.37}$$

while, as $\eta \to 0$,
$$\Theta_1 \to 4C\ln\tfrac{1}{2}\eta+(6C+\tfrac{1}{2}). \tag{2.38}$$

Integrating (2.29) and (2.30) subject to the boundary conditions (2.31), (2.32) leads to

$$\theta_1 = D-\ln\xi^2-\int_{\xi^2}^{\infty}\frac{e^{-t}}{t}\,dt, \tag{2.39}$$

$$g = -6\xi^2, \tag{2.40}$$

and so, as $\xi \to \infty$,
$$\theta_1 \to D-\ln\xi^2. \tag{2.41}$$

The three remaining arbitrary constants A, C and D are now determined by the matching, and found to be

$$C = -\tfrac{1}{2},\quad A = 1,\quad D = -\tfrac{5}{2}. \tag{2.42}$$

Hence $\Theta_1(\eta)$ and $\theta_1(\xi)$ are given by

$$\Theta_1 = -\frac{5}{2}\frac{1}{u^4}+\frac{1}{2}\left(\frac{1}{u^3}-\frac{3}{u^5}\right)\ln\frac{u-1}{u+1}, \tag{2.43}$$

$$\theta_1 = -\tfrac{5}{2}-\ln\xi^2-\int_{\xi^2}^{\infty}\frac{e^{-t}}{t}\,dt. \tag{2.44}$$

The perturbation temperature distribution in region I corresponds to a line heat source of strength

$$\begin{aligned} L(z) = -r\frac{\partial\theta}{\partial r}\bigg|_{\eta=0} &= -z^{-3}[\beta\eta\Theta_1'(\eta)]_{\eta=0}, \\ &= 2\beta z^{-3}, \end{aligned} \tag{2.45}$$

on $r = 0$, together with a continuous distribution of sinks throughout the fluid. The former results from outflow of heat from the jet by diffusion, and is related to the vertical transport of heat by convection in region II by

$$\int_0^{\xi(\to\infty)}\beta u_z\theta r\,dr = \beta z^{-2} = \int_z^{\infty}L(z)\,dz. \tag{2.46}$$

The heat sinks are provided by the inward convection of colder fluid from infinity, while the associated total vertical heat transport is

$$\int_{\eta(\to 0)}^{\infty}\beta u_z\theta r\,dr = -\tfrac{1}{4}\beta z^{-2}. \tag{2.47}$$

Finally, the vertical heat flux by diffusion,

$$-\int_0^\infty \frac{\partial \theta}{\partial z} r\,dr = -\tfrac{3}{4}\beta z^{-2}, \tag{2.48}$$

combines with the convective transport to give no total vertical heat flux ($F = 0$).

(ii) *The magnetic field.* As before it is supposed that

$$\chi = z^2(\tfrac{1}{2}\eta^2) + \gamma z \Phi_1(\eta) + \ldots, \tag{2.49}$$

in region I, and

$$\chi = z(2\xi^2) + \gamma \phi_1(\xi) + \ldots, \tag{2.50}$$

in region II.

On substituting (2.49) and (2.50) into (1.22), and retaining only the highest powers of z, we have

$$(1+\eta^2)\,\Phi_1'' - \eta^{-1}\Phi_1' = \begin{cases} \eta \Psi_0' & (z > 0), \\ 0 & (z < 0), \end{cases} \tag{2.51}$$

$$\xi(\xi^{-1}\phi_1')' = 2\xi\psi_0', \tag{2.52}$$

together with the boundary conditions

$$\phi_1 = (\xi^{-1}\phi_1')' = 0 \quad \text{at} \quad \xi = 0, \tag{2.53}$$

continuity of

$$\mathbf{b} \quad \text{on} \quad z = 0, \tag{2.54}$$

$$\Phi_1 = 0 \quad \text{at} \quad \eta = -0. \tag{2.55}$$

Integration of (2.51) is again facilitated by the substitution (2.34), which leads to

$$\frac{d^2\Phi_1}{du^2} = \begin{cases} -u^{-3} & (z > 0), \\ 0 & (z < 0). \end{cases} \tag{2.56}$$

Thus (2.52) and (2.56) have the general solutions

$$\Phi_1 = \begin{cases} -\tfrac{1}{2}u^{-1} + A + Bu & (z > 0), \\ C + Du & (z < 0), \end{cases} \tag{2.57}$$

$$\phi_1 = \xi^2 + e^{-\xi^2} + E\xi^2. \tag{2.58}$$

The matching conditions and boundary conditions then lead to

$$\left.\begin{aligned} A + B - \tfrac{1}{2} = 0, \quad \tfrac{1}{2} + B = 1 + E, \quad A = C, \\ B = D, \quad C = D \quad (\text{noting } u < 0, \text{ when } z < 0). \end{aligned}\right\} \tag{2.59}$$

Hence we have

$$\Phi_1 = \begin{cases} \dfrac{1}{4}\dfrac{(u+2)(u-1)}{u} & (z > 0), \\ \tfrac{1}{4}(u+1) & (z < 0), \end{cases} \tag{2.60}$$

$$\phi_1 = \tfrac{1}{4}(3\xi^2 + 4\,e^{-\xi^2}).\dagger \tag{2.61}$$

The resulting magnetic field lines in region I, are illustrated in figure 3.

† The solutions corresponding to (2.60) and (2.61) when the solid is a perfect conductor are

$$\Phi_1 = \begin{cases} \dfrac{u-1}{2u} & (z > 0), \\ 0 & (z < 0), \end{cases}$$

$$\phi_1 = \tfrac{1}{2}\xi^2 + e^{-\xi}.$$

(iii) *The velocity distribution.* Since the perturbation temperature distribution and magnetic field are now known, the corresponding correction to the fluid velocity can be found. In region I, we suppose that

$$\psi = \Psi_0(\eta) + z^{-1}\Psi_1(\eta) + \dots \tag{2.62}$$

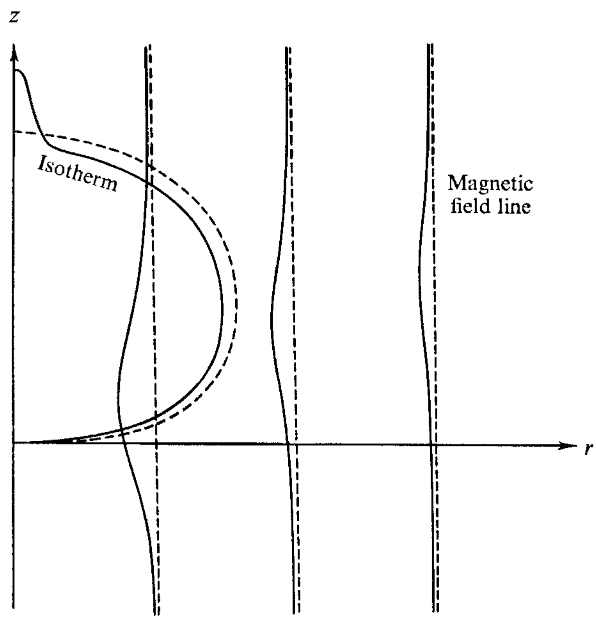

FIGURE 3. The perturbed magnetic field lines and isotherms.

After substituting into (1.21), and making the usual approximations, we find that Ψ_1 satisfies the equation

$$(\eta^2\Psi_1)'' = \beta\eta\Theta_1' - \gamma(2\Phi_1\Psi_0''' + 4\Phi_1'\Psi_0'), \tag{2.63}$$

together with the boundary conditions

$$\Psi_1 = O(\eta^{-1}) \quad \text{as} \quad \eta \to 0, \tag{2.64}$$

$$\Psi_1 = O(\eta^{-2}) \quad \text{as} \quad \eta \to \infty. \tag{2.65}$$

Integrating (2.63) subject to these boundary conditions leads to

$$\Psi_1 = \beta F_1(\eta) + \gamma F_2(\eta) + 2\frac{A(\beta,\gamma)}{\eta}, \tag{2.66}$$

where

$$\left.\begin{aligned} F_1(\eta) &= \eta^{-2}\int_0^\eta\left[\int_0^x y\Theta_1'(y)\,dy\right]dx,\\ F_2(\eta) &= -\eta^{-2}\int_0^\eta\left[\int_0^x (2\Phi_1\Psi_0''' + 4\Phi_1'\Psi_0')\,dy\right]dx,\\ A(\beta,\gamma) &= \tfrac{1}{32}\pi(2\beta-3\gamma).\dagger \end{aligned}\right\} \tag{2.67}$$

† The value of $A(\beta,\gamma)$ is evaluated by direct integration of

$$\int_0^\infty \eta\Theta_1'\,d\eta \quad \text{and} \quad \int_0^\infty (2\Phi_1\Psi_0''' + 4\Phi_1'\Psi_0')\,d\eta.$$

When the solid is a perfect conductor, $A(\beta,\gamma) = \frac{1}{32}\pi(2\beta-\gamma)$.

Moreover, as $\eta \to 0$, we have

$$\Psi_1 \sim \frac{2A}{\eta} - \beta + O(\eta). \tag{2.68}$$

Thus a matching can be made in region V with a solution in region II of the form,

$$\psi = \psi_0(\xi) + z^{-\frac{1}{2}} A \psi_1(\xi) + z^{-1} \psi_2(\xi) + \ldots, \tag{2.69}$$

provided that, as $\xi \to \infty$,

$$\psi_1 = \frac{1}{\xi} + O(\xi^{-2}), \tag{2.70}$$

$$\psi_2 \sim -\beta, \tag{2.71}$$

while the boundary conditions, on $r = 0$, imply that

$$\psi_1 = O(\xi^2), \quad \psi_2 = O(\xi^2). \tag{2.72}$$

Substituting into (1.21) we find that (for ψ_1) a balance of viscous and magnetic forces is maintained. Hence ψ_1 satisfies

$$\left[-\frac{3}{2} - \frac{\xi}{2}\frac{d}{d\xi} + \frac{\xi}{4}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)\right]\left[-\frac{1}{2} - \frac{\xi}{2}\frac{d}{d\xi} - \frac{\xi}{4}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)\right]\psi_1 = 0, \tag{2.73}$$

which has the solution

$$\psi_1 = \sqrt{\pi}\,\xi^2 e^{-\frac{1}{2}\xi^2}\{I_0(\tfrac{1}{2}\xi^2) - I_1(\tfrac{1}{2}\xi^2)\}, \tag{2.74}$$

satisfying the boundary conditions (2.70), (2.72), where I_n is the Bessel function of imaginary argument.

The problem for ψ_2 is very lengthy but straightforward. Forcing terms are produced by the convection and vertical diffusion of the lowest order θ, χ and ψ. Further, the solution is non-unique, because there is a non-zero complementary function of the equation which satisfies the condition $O(\xi^2)$, as $\xi \to 0$, and decays exponentially, as $\xi \to \infty$.

Consider the perturbation velocity distribution (see figure 4). In region I, the forcing terms resulting from the magnetic field and temperature distributions tend to make the fluid rise and fall, respectively. The perturbation velocity increases, as $\eta \to 0$, and in region V the motion is vertical, $\psi \sim 2A/r$ (the direction of flow depending on the sign of A). This ever increasing velocity is finally terminated by viscous action in a region II, where a large (compared to other perturbations) viscous eddy, $z^{-\frac{1}{2}}A\psi_1(\xi)$, is formed. Moreover, there is a critical value of the ratio γ/β for which there is no eddy, namely,

$$\frac{\gamma}{\beta} = \frac{2}{3} \quad (= 2 \text{ when the solid is a perfect conductor}). \tag{2.75}$$

It must be emphasized that the eddy is *not* a feature of the flow. The velocity corresponding to $z^{-\frac{1}{2}}A\psi_1(\xi)$ represents only a small correction to the basic flow.

To summarize, a self-consistent second approximation has now been obtained, where

$$\psi = \frac{z}{R}+\frac{2A}{r}+\frac{1}{z}\{\beta F_1(\eta)+\gamma F_2(\eta)\}, \tag{2.76}$$

$$\theta = \frac{z}{R^3}+\beta\left\{-\frac{5}{2}\frac{z}{R^4}+\frac{r^2-2z^2}{2R^5}\ln\frac{R-z}{R+z}\right\}, \tag{2.77}$$

$$\chi = \left\{\begin{array}{ll}\frac{1}{2}r^2+\frac{1}{4}\gamma(R+2z)(R-z)/R & (z>0),\\ \frac{1}{2}r^2+\frac{1}{4}\gamma(R+z) & (z<0),\dagger\end{array}\right\} \tag{2.78}$$

in region I, and

$$\psi = 1-e^{-\xi^2}+z^{-\frac{1}{2}}A\psi_1(\xi)+z^{-1}\psi_2(P,\beta,\gamma,\xi), \tag{2.79}$$

$$\theta = \frac{1}{z^2}-\frac{3}{2}\frac{r^2}{z^5}+\frac{\beta}{z^3}\left\{\ln z-\frac{5}{2}-\ln\xi^2-\int_{\xi^2}^{\infty}\frac{e^{-t}\,dt}{t}\right\}, \tag{2.80}$$

$$\chi = \tfrac{1}{2}r^2+\tfrac{1}{4}\gamma\{3\xi^2+4\,e^{-\xi^2}\}, \tag{2.81}$$

in region II.

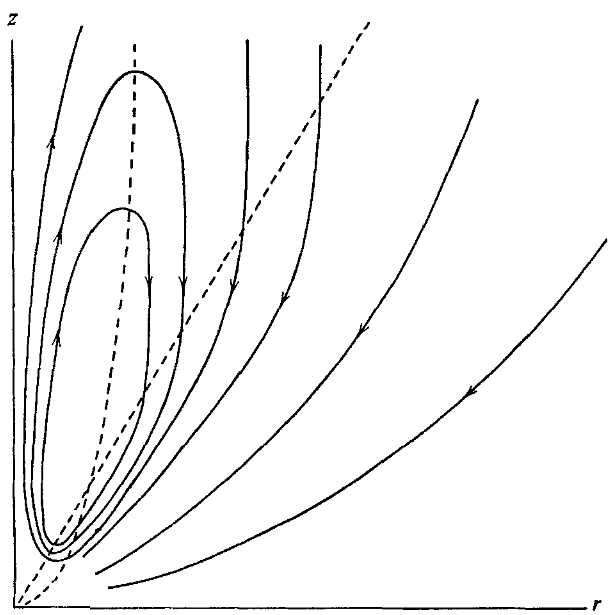

FIGURE 4. The perturbation streamlines for the stream function $z^{-\frac{1}{2}}A\psi_1(\xi)$, when $\gamma/\beta < \frac{2}{3}$. (Note that (i) the vertical streamlines in region V ultimately descend into region I, (ii) the eddy is not a feature of the full flow pattern.)

The above results show fairly conclusively that the model presented in § 2.1 is correct for $R \gg 1$, provided P, β, γ are of order 1. Moreover, as $R \to \infty$, the solutions will be valid whatever these values are. Finally, the strength of the lowest order corrections to ψ, θ and χ depend linearly on β and γ, while the influence of the Prandtl number first becomes apparent in the term $z^{-1}\psi_2(P,\beta,\gamma,\xi)$.

† When the solid is a perfect conductor (2.78) and (2.81) are replaced by

$$\chi = \left\{\begin{array}{ll}\frac{1}{2}r^2+\frac{1}{2}\gamma(z-z^2/R) & (z>0),\\ \frac{1}{2}r^2 & (z<0),\end{array}\right.$$

$$\chi = \tfrac{1}{2}r^2+\gamma(\tfrac{1}{2}\xi^2+e^{-\xi^2}).$$

3. The line source of heat

3.1. *The solution for $\epsilon_0 = \alpha_0 = 0$, $\epsilon_0 P^{-1}$ finite*

For the general case $\epsilon_0 \neq 0$, $\alpha_0 \neq 0$, the resulting flow is very different from the motion caused by a point heat source. In particular thermal convection cannot be ignored even for $\epsilon_0 \ll 1$. However, throughout § 3, the systematic approach of the previous section is adopted. The space $y > 0$ is divided into the four regions described by (2.23), where r, z are replaced by x, y respectively. Then arguments, similar to those of § 2, lead to the corresponding similarity variables,

$$\left.\begin{aligned} &\eta = x/y \quad \text{in region I,} \\ \text{and} \qquad &\xi = x/2y^{\frac{1}{2}} \quad \text{in region II.} \end{aligned}\right\} \tag{3.1}$$

Before becoming involved in the general case, it is useful to know the solution to the simplified problem $\epsilon_0 = \alpha_0 = 0$, $\epsilon_0 P^{-1}$ finite. For this particular case, corresponding to an infinite thermal and magnetic diffusivity, there can be no convection of heat or of the magnetic field lines. Hence, (1.35) and (1.36) become

$$\nabla^2\theta = \nabla^2\chi = 0. \tag{3.2}$$

The solutions of these equations satisfying the boundary conditions (1.39)–(1.41) are:

$$\theta = \frac{1}{\pi}\frac{y}{x^2+y^2}, \tag{3.3}$$

$$\chi = -x. \tag{3.4}$$

Hence in region I, θ is given by

$$\theta = y^{-1}\Theta_0(\eta) = y^{-1}(1/\pi)(1+\eta^2)^{-1}, \tag{3.5}$$

while in region II

$$\theta = y^{-1}\frac{1}{\pi} - y^{-2}\frac{4}{\pi}\xi^2 + \dots. \tag{3.6}$$

Consideration of (1.34) indicates that the required form for ψ is

$$\left.\begin{aligned} &\psi = \Psi_0(\eta) \quad \text{in region I,} \\ \text{and} \qquad &\psi = \psi_0(\xi) \quad \text{in region II.} \end{aligned}\right\} \tag{3.7}$$

Substituting into (1.34), and retaining only the highest powers of y, leads to

$$\Theta_0' + (\eta^2\Psi_0')' = 0, \tag{3.8}$$

$$\tfrac{1}{4}\psi_0^{\text{iv}} - (\xi^2\psi_0'' + 3\xi\psi_0') = 0. \tag{3.9}$$

These equations are solved subject to the boundary conditions,

$$\psi_0 = \psi_0'' = 0 \quad \text{on} \quad \xi = 0, \tag{3.10}$$

$$\Psi_0 = O(\eta^{-1}) \quad \text{as} \quad \eta \to \infty, \tag{3.11}$$

and the matching conditions in region V. Hence we have

$$\Psi_0 = (1/\pi)\tan^{-1}\eta - \tfrac{1}{2}, \tag{3.12}$$

$$\psi_0 = -\tfrac{1}{2}\operatorname{erf}\xi, \tag{3.13}$$

where ψ_0 is the solution proposed by Jungclaus (1965) for the two-dimensional jet. The pressure distribution

$$p(x,y) = \frac{1}{\pi}\ln(x^2+y^2)^{\frac{1}{2}} + y^{-\frac{1}{2}}\frac{1}{2\sqrt{\pi}}e^{-\xi^2}, \tag{3.14}$$

for $y \gg 1$, is now determined by integrating the equation of motion.

The solution shows similar features to the flow above the point source of heat in § 2.1. Fluid is forced radially inwards from the outer regions and is ejected along the axis $x = 0$. Moreover, as both problems are linear, the solution in this section may be regarded as the superposition of flows, resulting from point sources of heat distributed along the line $(0, 0)$.

3.2. *The general problem*

The analysis in the previous section suggests that a solution may exist in region I of the form,

$$\psi = \Psi_0(\eta), \quad \theta = y^{-1}\Theta_0(\eta), \quad \chi = y\Phi(\eta). \tag{3.15}$$

Substituting these values into (1.34)–(1.36), and retaining only the terms containing the highest powers of y, leads to

$$\Theta_0' + (\Phi^2\Psi_0')' = 0, \tag{3.16}$$

$$(1+\eta^2)\,\Phi'' = -\alpha_0\Psi_0'\,\Phi, \tag{3.17}$$

$$[(1+\eta^2)\,\Theta_0]'' = \epsilon_0\Psi_0'\,\Theta_0. \tag{3.18}$$

Moreover, the last two equations are exact as no terms have been neglected. In terms of the similarity variables, the conditions (1.39)–(1.41) become

$$\Psi_0 = O(\eta^{-1}), \quad \Theta_0 = O(\eta^{-2}), \quad \Phi = O(\eta) \quad \text{as} \quad \eta \to \infty,\dagger \tag{3.19}$$

$$\Psi_0 = o(\eta^{-1}), \quad \Phi \sim -\eta \quad \text{as} \quad \eta \to 0, \tag{3.20}$$

and

$$\int_0^\infty \Theta_0\,d\eta = \tfrac{1}{2}. \tag{3.21}$$

The condition $\Psi_0 = o(\eta^{-1})$ as $\eta \to 0$ is equivalent to demanding that the horizontal velocity should vanish as $y \to \infty$, for fixed x. Integrating (3.16) subject to the boundary condition (3.20) leads to

$$\Phi^2\Psi_0' = \Theta_0(0) - \Theta_0(\eta). \tag{3.22}$$

Moreover, integration of the governing equations gives the pressure distribution,

$$p(x,y) = \Theta_0(0)\ln y + \int^\eta \Psi_0'\,\Phi\Phi'\,d\eta. \tag{3.23}$$

At this stage, it is convenient to renormalize the problem by the change of variables,

$$\left.\begin{aligned} \Psi(\eta) &= \Psi_0(\eta)/\Theta_0(0), \quad & \Theta(\eta) &= \Theta_0(\eta)/\Theta_0(0), \\ \epsilon &= \Theta_0(0)\,\epsilon_0, & \alpha &= \Theta_0(0)\,\alpha_0, \end{aligned}\right\} \tag{3.24}$$

† These three boundary conditions are not independent. The conditions on Ψ_0 and Φ are determined by (3.17), (3.18), (3.22) and the boundary condition on Θ_0.

where $\Theta_0(0)$ is determined subsequently from (3.21) as

$$\Theta_0(0) = \left[2\int_0^\infty \Theta(\eta)\,d\eta\right]^{-1}. \tag{3.25}$$

Hence (3.22), (3.17) and (3.18) become

$$\Phi^2\Psi' = 1-\Theta, \tag{3.26}$$

$$(1+\eta^2)\,\Phi'' = -\alpha\Psi'\Phi, \tag{3.27}$$

$$[(1+\eta^2)\,\Theta]'' = \epsilon\Psi'\Theta. \tag{3.28}$$

Consider the solutions of these equations as $\eta \to 0$. From the boundary condition (3.20) we have $\Phi \sim -\eta$, so that (3.26) and (3.28) lead to

$$\eta^2 f'' + \epsilon f = 0 \quad \text{as} \quad \eta \to 0, \tag{3.29}$$

where the function f is defined by

$$\Theta(\eta) = \frac{1-f(\eta)}{1+\eta^2}. \tag{3.30}$$

Hence, f is given by
$$f \sim A_1\eta^{n_1} + A_2\eta^{n_2}, \tag{3.31}$$

where n_1 and n_2 are roots of the equation,

$$n(n-1)+\epsilon = 0. \tag{3.32}$$

Moreover, n_1 and n_2 are real only if $\epsilon < \frac{1}{4}$ and so, if $\epsilon > \frac{1}{4}$,

$$f \sim \eta^{\frac{1}{2}}(A\eta^{\nu} + \bar{A}\eta^{-\nu}), \tag{3.33}$$

where A is a complex constant and $\nu = n_1 - \frac{1}{2}$ (pure imaginary).

For the present, attention is restricted to the case $\epsilon < \frac{1}{4}$. Now since we do not know if either A_1 or A_2 is to be zero, we put

$$f \sim A\eta^n \quad \text{as} \quad \eta \to 0. \tag{3.34}$$

Hence we have
$$\Psi \sim \frac{A}{n-1}\eta^{n-1}, \quad \Theta \sim 1 - A\eta^n, \quad \Phi \sim -\eta. \tag{3.35}$$

It follows that the conditions on the inner problem, as $\xi \to \infty$, are:

$$\psi \sim y^{\frac{1}{2}-\frac{1}{2}n}\left(A\frac{2^{n-1}}{n-1}\xi^{n-1}\right), \tag{3.36}$$

$$\theta \sim y^{-1} - y^{-1-\frac{1}{2}n}(A2^n\xi^n), \tag{3.37}$$

$$\phi \sim -y^{\frac{1}{2}}(2\xi). \tag{3.38}$$

(i) *The inner problem.* Restricting attention to the case $\epsilon < \frac{1}{4}$, the asymptotic forms (3.36)–(3.38) require that the solution in region II should be of the form,

$$\psi^* = y^{\frac{1}{2}-\frac{1}{2}n}\psi_0(\xi), \tag{3.39}$$

$$\theta^* = y^{-1} + y^{-1-\frac{1}{2}n}\theta_0(\xi), \tag{3.40}$$

$$\chi = y^{\frac{1}{2}}\phi_0(\xi), \tag{3.41}$$

where the change of variables,

$$\psi = \Theta_0(0)\,\psi^*, \quad \theta = \Theta_0(0)\,\theta^*, \tag{3.42}$$

has been made. Substituting into (1.35), and retaining only the highest power of y, we obtain

$$\phi_0'' = 0, \tag{3.43}$$

so that the uniform magnetic field given by

$$\chi = -y^{\frac{1}{2}}(2\xi), \tag{3.44}$$

is maintained. Thus, repeating the procedure in (1.34) and (1.36) leads to

$$0 = 2\theta_0' + \{(n^2-1)\,\psi_0 + (2n+1)\,\xi\psi_0' + \xi^2\psi_0''\} - \tfrac{1}{4}\psi_0^{\text{iv}}, \tag{3.45}$$

$$\theta_0'' = 2\epsilon\psi_0', \tag{3.46}$$

together with the boundary conditions

$$\theta_0' = 0, \quad \psi_0 = \psi_0'' = 0 \quad \text{at} \quad \xi = 0, \tag{3.47}$$

$$\psi_0 \sim A\frac{2^{n-1}}{n-1}\xi^{n-1} \quad \text{as} \quad \xi \to \infty, \tag{3.48}$$

where the condition (3.48) is sufficient to satisfy both the boundary conditions (3.36) and (3.37). Integrating (3.46) leads to

$$\theta_0(\xi) = C + 2\epsilon\int_0^\xi \psi_0(x)\,dx. \tag{3.49}$$

The constant C remains undetermined as to the present order it is negligible in the matching. After substitution for θ_0, (3.45) becomes

$$\tfrac{1}{4}\psi_0^{\text{iv}} - \xi^2\psi_0'' - (2n+1)\,\xi\psi_0' + (3n^2-4n+1)\,\psi_0 = 0. \tag{3.50}$$

For $n < \frac{1}{2}$, this equation has no solution satisfying the boundary conditions, while, for $n > \frac{1}{2}$, the solution is given (see appendix A) by

$$\psi_0 = A\frac{2^{\frac{3}{2}-2n}}{\pi}\frac{\Gamma(\frac{3}{2}-n)\cos n\pi}{\Gamma(2-n)\cos\frac{1}{2}n\pi}\int_0^\infty p^{n-1}K_{n-\frac{1}{2}}\left(\frac{p^2}{4}\right)\sin p\xi\,dp. \tag{3.51}$$

Hence, matching the solutions in regions I and II in region V determines the remaining boundary condition to be imposed on the outer solution, namely,

$$\left.\begin{aligned} &A_2 = 0,\\ \text{or}\qquad &f/\eta^{n_1} \to \text{constant}\,(=A_1 = A),\end{aligned}\right\} \tag{3.52}$$

where $n_1 > \frac{1}{2} > n_2$ and A_1, A_2 are defined by (3.31). Finally, solving the outer problem determines the constant A.

When $\epsilon > \frac{1}{4}$, a matching might be obtained if

$$\psi = y^{\frac{1}{2}}\{y^{\nu}\psi_0 + y^{-\nu}\overline{\psi}_0\}. \tag{3.53}$$

However, with ψ of order $y^{\frac{1}{2}}$, the inertia term in the equation of motion is comparable with the terms already retained. Thus, the inner problem becomes nonlinear and it is no longer possible for ψ to be represented by (3.53). Even if

ϵP^{-1} were sufficiently small for inertia to be neglected, no solution could be obtained. Essentially, ψ_0 would still be of the form (3.51) and, as $\xi \to \infty$,

$$\psi_0 \sim \xi^{-\frac{1}{2}}(A\xi^{\nu}+B\xi^{-3\nu}). \tag{3.54}$$

Unfortunately this solution cannot be equated to the solution (3.33) in the region of overlap.

(ii) *The outer problem* $\epsilon < \frac{1}{4}$, $\alpha = 0$. By considering the motion when the magnetic field lines are unperturbed ($\alpha = 0$), solutions of the outer problem are obtained for $0 \leqslant \epsilon < \frac{1}{4}$. Moreover, it will become apparent, from considering the case $\epsilon = 0$, $\alpha \ll 1$, that an approximately uniform magnetic field can be maintained and hence that the results presented here for $\alpha = 0$ are still valid when $\alpha \ll 1$.

Substituting $\Phi = -\eta$, (3.26) and (3.28) become

$$\Psi' = (1-\Theta)/\eta^2, \tag{3.55}$$

$$[(1+\eta^2)\Theta]'' = \epsilon\Theta(1-\Theta)/\eta^2, \tag{3.56}$$

together with the boundary conditions,

$$\Theta \sim 1-A(\epsilon)\eta^{n_1} \quad \text{as} \quad \eta \to 0, \tag{3.57}$$

$$\Theta = O(\eta^{-2}) \quad \text{as} \quad \eta \to \infty, \tag{3.58}$$

where $n_1(> n_2)$ is given by (3.32), and $A(\epsilon)$ is a constant to be determined. The boundary condition, as $\eta \to 0$, has been imposed by the form of the solution in region II.

We propose the following series solution

$$\Theta = \sum_{n=0}^{\infty} \epsilon^n \Theta_n(\eta), \tag{3.59}$$

where
$$\Theta_0 = 1/(1+\eta^2), \tag{3.60}$$

$$\Theta_1 = -\tfrac{1}{2}\eta(\tfrac{1}{2}\pi - \tan^{-1}\eta)/(1+\eta^2), \tag{3.61}$$

and
$$\Theta_n = \frac{-1}{1+\eta^2}\int_0^{\eta}\left[\int_x^{\infty}\left\{\frac{\Theta_{n-1}}{y^2} - \sum_{r=0}^{n-1}\frac{\Theta_{n-r-1}\Theta_r}{y^2}\right\}dy\right]dx \quad (n \geqslant 1). \tag{3.62}$$

This result is obtained formally by substituting (3.59) into (3.56) and equating powers of ϵ. In order to establish that the boundary conditions are satisfied, we assume that for $n = 1, 2, \ldots, N-1$, Θ_n takes the asymptotic forms,

$$\Theta_n \sim -\tfrac{1}{4}\pi\eta(-\ln\eta)^{n-1}/(n-1)! \quad (n \geqslant 1) \quad \text{as} \quad \eta \to 0, \tag{3.63}$$

$$\Theta_n = O(\eta^{-2}) \quad \text{as} \quad \eta \to \infty, \tag{3.64}$$

and that Θ_n is bounded for $0 \leqslant \eta \leqslant \infty$. Clearly, applying these conditions to (3.62) indicates that Θ_N also satisfies them. It follows that (3.63) and (3.64) hold for all values of n. Moreover, summing the expressions for Θ_n, as $\eta \to 0$, gives

$$\Theta \sim 1-\epsilon(\tfrac{1}{4}\pi)\eta^{1-\epsilon}. \tag{3.65}$$

Now, from (3.32), we have
$$\left.\begin{aligned} n_1 &= 1-\epsilon+O(\epsilon^2), \\ n_2 &= \epsilon+O(\epsilon^2), \end{aligned}\right\} \tag{3.66}$$
and hence the asymptotic form (3.65) satisfies the boundary condition (3.57).

Unfortunately, though we have been able to show that the solution (3.59) converges to the correct asymptotic form, as $\eta \to 0$, we have been unable to establish that the series converges for all η. However (3.56) was also solved numerically (details are given in appendix B) on the Cambridge University Titan computer for various values of ϵ (see figure 5). The values of $A(\epsilon)$ and $f(\infty)$ (as defined by (3.30)) were found to be in close agreement with the approximate values,
$$A(\epsilon) = (\tfrac{1}{4}\pi)\,\epsilon, \quad f(\infty) = \tfrac{1}{2}\epsilon, \tag{3.67}$$
determined by (3.65) and (3.61) (see table 1).

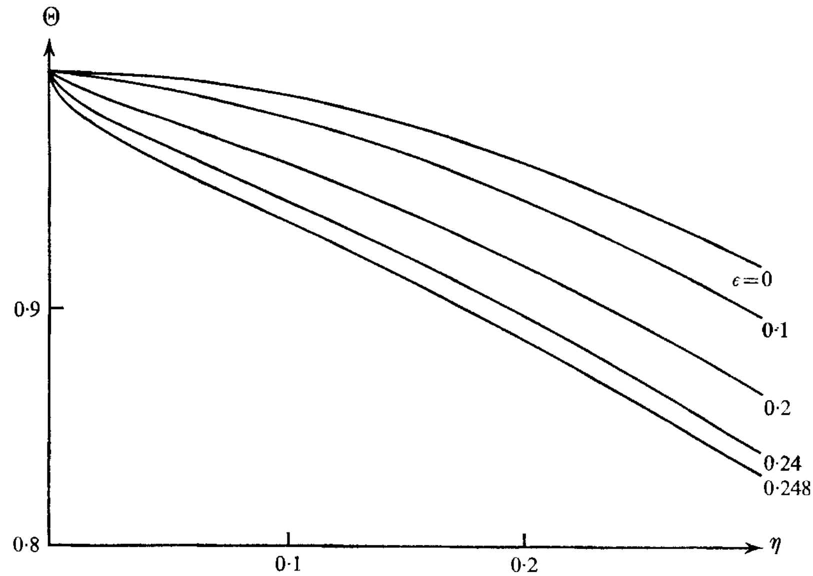

FIGURE 5. The computed Θ curves for various values of $\epsilon(<0{\cdot}25)$ near $\eta = 0$.

When $\epsilon \leqslant 1$, the outer solution near $\eta = 0$ is
$$\Theta \sim 1-(\tfrac{1}{4}\pi)\,\epsilon\eta^{1-\epsilon}+\ldots, \tag{3.68}$$
$$\Psi \sim -(\tfrac{1}{4}\pi)\,\eta^{-\epsilon}+\ldots, \tag{3.69}$$
while, for $\eta = O(1)$,
$$\Theta = 1/(1+\eta^2), \tag{3.70}$$
$$\Psi = \tan^{-1}\eta-(\tfrac{1}{2}\pi). \tag{3.71}$$
Evidently, by considering (3.55), (3.56), the limiting process η fixed, $\epsilon \to 0$ followed by $\eta \to 0$ will yield the result,
$$\Psi \to -\tfrac{1}{2}\pi, \tag{3.72}$$
in (3.69) (in agreement with (3.71)). However, *erroneously* truncating the series (3.69) after the first term, and taking the limit $\epsilon \to 0$, leads to $\Psi \to -\tfrac{1}{4}\pi$. Hence,

for $\epsilon \ll 1$, we have two asymptotic solutions valid in region II. For $y^{\frac{1}{2}\epsilon} \doteqdot 1$, the analysis of § 3.1 is valid giving the solution,

$$\psi_0 = -(\tfrac{1}{2}\pi)\,\mathrm{erf}\,\xi, \tag{3.73}$$

while, for $y^{\frac{1}{2}\epsilon} \gg 1$, the analysis above indicates that

$$\psi_0 = -(\tfrac{1}{4}\pi)\,y^{\frac{1}{2}\epsilon}\,\mathrm{erf}\,\xi \quad (\xi = O(1)). \tag{3.74}$$

	α	A	$f(\infty)$	$b_y(x, 0)$	$-b_x(\lvert x\rvert, 0)$
$\epsilon = 0$, $n = 1$	0·1			0·949	0·08
	0·2			0·897	0·16
	0·4			0·787	0·35
	0·6			0·670	0·56
	0·8	0	0	0·542	0·82
	1·0			0·400	1·14
	1·2			0·236	1·62
	1·3			0·140	2·01
	1·4			0·021	3·06
$\epsilon = 0{\cdot}1$, $n = 0{\cdot}8873$	0	0·0773	0·0531	1	0
$\epsilon = 0{\cdot}2$, $n = 0{\cdot}7236$	0	0·1528	0·1178	1	0
	0·1	0·157	0·123	0·934	0·09
	0·2	0·163	0·130	0·863	0·18
	0·4	0·176	0·145	0·707	0·40
	0·6	0·194	0·169	0·517	0·70
	0·8	0·225	0·219	0·253	1·24
	0·85	0·238	0·247	0·155	1 53
$\epsilon = 0{\cdot}24$, $n = 0{\cdot}6$	0	0·1813	0·1550	1	0
$\epsilon = 0{\cdot}248$, $n = 0{\cdot}5447$	0	0·185	0·1674	1	0
	0·1	0·196	0·180	0·920	0·09
	0·2	0·212	0·194	0·830	0·19
	0·4	0·263	0·242	0·597	0·47
	0·45	0·286	0·264	0·516	0·57
	0·5	0·319	0·298	0·411	0·72
	0·525	0·344	0·325	0·339	0·83
	0·55	0·387	0·375	0·228	1·02
	0·56	0·424	0·427	0·137	1·24

TABLE 1. The computed values of A, $f(\infty)$, $b_y(x, 0)$, $-b_x(|x|, 0)$ for various values of ϵ and α. For $\alpha = 0$, the figures are accurate to within $\frac{1}{2}$ %.

Clearly, the motion for $\epsilon \neq 0$ is very different from the simple solution described in § 3.1. Fluid is still driven radially inwards towards the origin in region I and ejected along the axis $x = 0$, but superimposed on this motion is an eddy in region II whose flux increases as $y^{\frac{1}{2}-\frac{1}{2}n}$ (figures 6 and 7). Since the main bulk of the flow is in region II, as $y \to \infty$, it might be anticipated that the motion would not depend on the flow in region I. This is not the case, as the flow in region II is not unique and depends on the matching made in region V. The buoyancy torque in region II, which gives rise to the eddy, results from small disturbances of the basic temperature distribution caused by the convection of heat by the induced motion, i.e. to lowest order $\theta = y^{-1}$, $\partial\theta/\partial x = 0$. The situation resembles to some extent linearized Benard convection. Moreover, this suggests that, when $\epsilon > \frac{1}{4}$, the motion will consist of numerous eddies (owing to to the unstable stratification) and hinted at by (3.33) whose structure will depend on viscosity (and possibly inertia), or even that no steady solution is possible (cf. discussion of the magnetic field in § 3.3).

(iii) *The outer problem* $\epsilon < \frac{1}{4}$, $\alpha \neq 0$. The full non-linear equations (3.26)–(3.28) were integrated numerically for various values of ϵ, and α on the Titan computer. The computed values of A_1 (equation (3.31), $A_2 = 0$), $f(\infty)$ (equation (3.30)), $b_y(x, 0)$, $b_x(|x|, 0)$ are given in table 1. The details of the calculation are given in appendix B. However, it is worth noting here that, as $\eta \to \infty$, (3.27) has the asymptotic solution,

$$\Phi = -b_y(x, 0)\,\eta - b_x(|x|, 0) + O(\eta^{-1}), \tag{3.75}$$

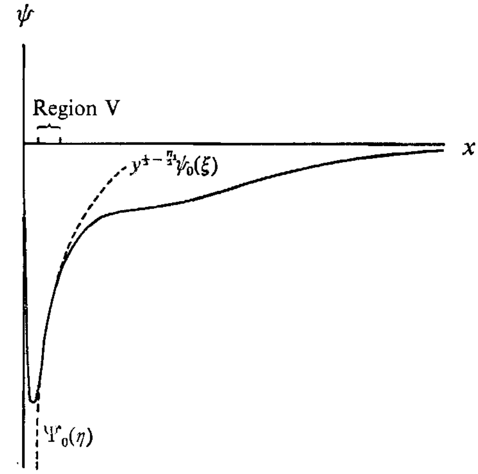

FIGURE 6. The stream function, for fixed y. (Note the region of overlap.)

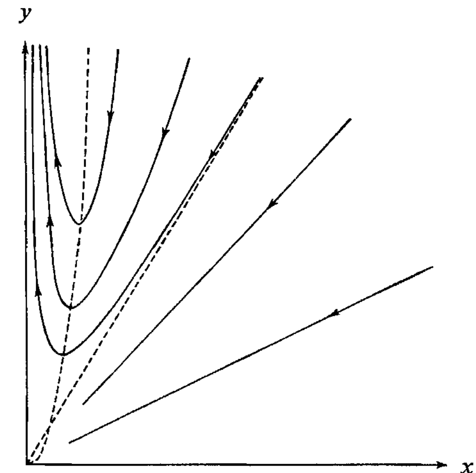

FIGURE 7. The streamlines in regions I and II, for $\epsilon < 0{\cdot}25$.

where $b_y(x, 0)$, $b_x(|x|, 0)$, (the vertical and horizontal components of the magnetic field on $x > 0$, $y = 0$) are both constants. Moreover, the numerical results indicate that (for given ϵ) the value of $b_y(x, 0)$ decreases monotonically from 1 to 0 as α increases from 0 to $\overline{\alpha}(\epsilon)$ (some constant depending on the value of ϵ), i.e. the analysis is valid only for $0 \leqslant \alpha \leqslant \overline{\alpha}(\epsilon)$.

For the particular case $\epsilon = 0$ (corresponding to an infinite thermal conductivity), a series solution for Φ may be obtained when α is small. Equation (3.28) has the solution

$$\Theta = (1+\eta^2)^{-1}, \tag{3.76}$$

and (3.26), (3.27) reduce to

$$\Phi^2\Psi'' = \eta^2/(1+\eta^2), \tag{3.77}$$

$$\Phi\Phi'' = -\alpha\eta^2/(1+\eta^2)^2. \tag{3.78}$$

Hence, for $\alpha \ll 1$, we have the solution

$$\Phi = -\eta + \tfrac{1}{2}\alpha(\eta - \tan^{-1}\eta) + O(\alpha^2). \tag{3.79}$$

Clearly, to order α, the magnetic field is

$$\mathbf{b} = \begin{cases} [\tfrac{1}{2}\alpha(\eta/(1+\eta^2) - \tan^{-1}\eta), \quad 1 - \tfrac{1}{2}\alpha\eta^2/(1+\eta^2)] \quad (y > 0), & (3.80) \\ [0, \quad 1 - \tfrac{1}{2}\alpha] \quad (y < 0), & (3.81) \end{cases}$$

and in particular, on $y = 0$,

$$\mathbf{b}(x, 0) = [-(\operatorname{sgn} x)\tfrac{1}{4}\pi\alpha, \quad 1 - \tfrac{1}{2}\alpha]. \tag{3.82}$$

Finally, the magnetic field lines for $\alpha \neq 0$ are illustrated in figure 8.

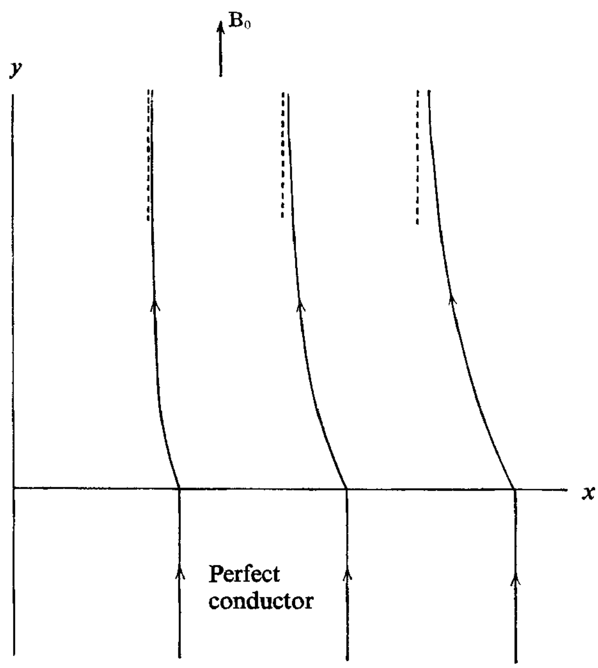

FIGURE 8. The perturbed magnetic field lines, when $\alpha \neq 0$.

3.3. *Discussion*

The difference between the point heat source and line heat source problems is interesting. Evidently, the difference results from the importance, in the case of the line heat source, of thermal convection (and to a lesser extent advection of the magnetic field). Mathematically, the reasons for this are clear from the form of the similarity solutions. However, physically, the larger input of heat at the origin in the case of the line heat source, results in a larger flux of fluid, and it is for this reason that advective effects are now important.

The choice of boundary conditions on the magnetic field for the line heat source problem requires discussion. Clearly, the choice of a perfectly conducting solid makes the problem well posed. In this case, a current sheet flows at the surface of the solid, which balances to some extent the effect of current produced in the fluid. If we suppose that the solid is of finite conductivity, and that a radial inflow of the type described in § 3.1 is maintained (e.g. assuming $\epsilon = 0$, $0 < \alpha \ll 1$), then the currents produced by the motion are such as totally disrupt the conditions of a uniformly applied magnetic field at infinity. Moreover, if we were to switch on the temperature distribution at time $t = 0$ and consider the transient problem, it is likely that the resulting motions will never settle down to a steady state as the uniform magnetic field at a great distance is likely to be ultimately disturbed. To illustrate this point, we consider the following time dependent problem, which resembles to some extent the speculated motion when $\epsilon = 0$, $0 < \alpha \ll 1$. At time $t = 0$, an unbounded fluid is permeated by a uniform magnetic field $\mathbf{b} = \hat{\mathbf{y}}$. Subsequently, a radial sink flow with velocity $u_r = -1/r\,(r = (x^2+y^2)^{\frac{1}{2}})$ is maintained, while the applied magnetic field at infinity is kept constant $\mathbf{b} = \hat{\mathbf{y}}$. The resulting magnetic field may be expressed in terms of the magnetic vector potential χ in the similarity form

$$\chi = -xF(r^2/4t), \tag{3.83}$$

where F and the solution of this problem are given in appendix C. Clearly, there is no ultimate steady state. Thus, it is in this sense that we may speculate that there is no steady solution to the line heat source problem when the solid is of finite conductivity. However, if the fluid and solid are of finite extent (as they must be in a physical situation), it is to be expected that for α sufficiently small an approximately uniform magnetic field may be maintained, such that the analysis in § 3.2 is relevant in certain regions of flow.

When the solid is a perfect conductor, the problem is perhaps best considered as the ultimate steady state of a time dependent problem. Thus, initially a uniform magnetic field is maintained. Subsequently, the magnetic field is 'frozen' into the solid, and the magnetic field, as $y \to \infty$, ultimately settles down to a uniform vertical magnetic field with a value that is greater than its initial value. Thus, strictly, it is the magnetic field in the solid which is given, not the magnetic field, as $y \to \infty$. Clearly, from this point of view, the restriction on the values of α in § 3.2 (iii) is purely mathematical and is not a restriction on any of the initial values of physical quantities.

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Appendix A

In order to solve (3.50) subject to the boundary conditions (3.47) and (3.48) particular integrals are sought in the form

$$\psi(\xi) = \int_C F(p)\, e^{p\xi}\, dp, \tag{A 1}$$

where C is a contour in the complex plane to be chosen later. Substituting this value of ψ into (3.50), it is apparent that $F(p)$ must satisfy

$$(p^2F)'' - (2n+1)(pF)' - (3n^2-4n+1+\tfrac{1}{4}p^4)\,F = 0, \tag{A 2}$$

and the boundary condition

$$[\{\xi p^2F + (2n+1)\,pF - (p^2F)'\}\,e^{p\xi}]_c = 0. \tag{A 3}$$

Putting

$$F(p) = p^{n-1}G(p), \tag{A 4}$$

it follows that $G(p)$ satisfies

$$p(pG')' - [(2n-1)^2 + \tfrac{1}{4}p^4]\,G = 0. \tag{A 5}$$

Hence, $F(p)$ has the general solution,

$$F(p) = p^{n-1}\{AI_\nu(\tfrac{1}{4}p^2) + BI_{-\nu}(\tfrac{1}{4}p^2)\}, \tag{A 6}$$

where

$$\nu = n - \tfrac{1}{2}, \tag{A 7}$$

and I_ν is the Bessel function of imaginary argument.

There are four particular integrals corresponding to

$$\left.\begin{array}{ll} C_1\text{:} & p \text{ varies from } 0 \text{ to } \infty, \\ C_2\text{:} & p \text{ varies from } 0 \text{ to } -\infty, \end{array}\right\} \quad F(p) = p^{n-1}K_\nu(\tfrac{1}{4}p^2),$$

$$\left.\begin{array}{ll} C_3\text{:} & p \text{ varies from } 0 \text{ to } i\infty, \\ C_4\text{:} & p \text{ varies from } 0 \text{ to } -i\infty, \end{array}\right\} \quad F(p) = p^{n-1}K_\nu(-\tfrac{1}{4}p^2),$$

where K_ν is defined by

$$K_\nu(z) = \tfrac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \tag{A 8}$$

with the particular property

$$z^{\frac{1}{2}}\, e^z\, K_\nu(z) \to \text{constant} \quad \text{as} \quad z \to \infty \tag{A 9}$$

(Watson 1958, p. 202). In order to satisfy the boundary conditions at $\xi = 0$ an odd combination of particular integrals must be taken, pairing either C_1, C_2 or C_3, C_4. The former gives a divergent solution as $\xi \to \infty$, so the required solution must be given by

$$\psi(\xi) = \int_0^\infty p^{n-1}K_\nu(\tfrac{1}{4}p^2)\sin p\xi\, dp. \tag{A 10}$$

The boundary condition (3.48) can be satisfied only for $n > \frac{1}{2}$, in which case

$$\psi(\xi) \sim -2^{3n-\frac{5}{2}}\pi \frac{\Gamma(1-n)\cos(\frac{1}{2}n\pi)}{\Gamma(\frac{3}{2}-n)\cos n\pi}\,\xi^{n-1} \quad \text{as} \quad \xi \to \infty. \tag{A 11}$$

Hence, for $n < \frac{1}{2}$, there is no solution, while, for $n > \frac{1}{2}$,

$$\psi_0 = A\,\frac{2^{\frac{3}{2}-2n}}{\pi}\,\frac{\Gamma(\frac{3}{2}-n)\cos n\pi}{\Gamma(2-n)\cos(\frac{1}{2}n\pi)} \int_0^\infty p^{n-1}K_{n-\frac{1}{2}}(\tfrac{1}{4}p^2)\sin p\xi\, dp. \tag{A 12}$$

Appendix B

The method used to integrate numerically (3.26)–(3.28) subject to the boundary conditions,

$$\Phi = 0, \quad \Phi' = -1 \quad \text{at} \quad \eta = 0, \tag{B 1}$$

$$f \sim \begin{cases} A(\epsilon, \alpha)\,\eta^{n_1} & (\eta \to 0), \\ \text{constant} & (\eta \to \infty), \end{cases} \tag{B 2}$$

where f is defined by (3.30), is outlined.

Introducing three new functions $g(\eta)$, $p(\eta)$, $q(\eta)$, where

$$\Phi = -p\eta, \tag{B 3}$$

the governing equations may be expressed in the form,

$$\eta q' = -q - \alpha\frac{\eta^2+f}{p(1+\eta^2)^2}, \tag{B 4}$$

$$\eta p' = q, \tag{B 5}$$

$$\eta g' = g - \epsilon\frac{(\eta^2+f)(1-f)}{p^2(1+\eta^2)^2}, \tag{B 6}$$

$$\eta f' = g. \tag{B 7}$$

In order to avoid difficulties near the origin, we define the functions G_1 and G_2 as

$$G_1 = \frac{g-n_2 f}{(n_1-n_2)\,\eta^{n_1}}, \quad G_2 = \frac{g-n_1 f}{(n_2-n_1)\,\eta^{n_2}}, \tag{B 8}$$

which then satisfy

$$\left.\begin{aligned} \eta G_1' &= -\frac{\epsilon}{n_1-n_2}\frac{\eta^2+(1-p^2)f-(2p^2+1)\,\eta^2 f-p^2\eta^4 f-f^2}{p^2(1+\eta^2)^2\,\eta^{n_1}},\\ \eta G_2' &= -\frac{\epsilon}{n_1-n_2}\frac{\eta^2+(1-p^2)f-(2p^2+1)\,\eta^2 f-p^2\eta^4 f-f^2}{p^2(1+\eta^2)^2\,\eta^{n_2}},\end{aligned}\right\} \tag{B 9}$$

where

$$f = \eta^{n_1}G_1 - \eta^{n_2}G_2. \tag{B 10}$$

In terms of the new variables, the boundary conditions become

$$p(0) = 1, \quad q(0) = G_2(0) = g(\infty) = 0. \tag{B 11}$$

Equations (B 3)–(B 10) were integrated numerically using the Runge–Kutta–Gill method with a variable step length,

$$\delta = 0{\cdot}005\,\eta. \tag{B 12}$$

The integration was divided into two parts. For $\eta > 0{\cdot}1$, (B 4)–(B 7) were used while, for $\eta < 0{\cdot}1$, the integration was changed to (B 4), (B 5), (B 9) and (B 10).

For $\alpha = 0$, the value of $f(\infty)$ was adjusted so that, when integrating from ∞ to 0, $G_2(0) = 0$. A similar procedure was adopted when integrating from 0 to ∞ (for numerical purposes 0 and ∞ were taken at $\eta = O(10^{-3})$ and $O(10)$ respectively). An agreement of at least $\frac{1}{2}\,\%$ was achieved between the values of $f(\infty)$ and $G_1(0)$ $(= A(\epsilon, 0))$ obtained by integration in the different directions.

For $\alpha \neq 0$, the equations were integrated from 0 to ∞ only (for numerical purposes 0 and ∞ were taken at $O(10^{-5})$ and $O(10^3)$). No check of the accuracy was obtained by integration from ∞ to 0 due to the difficulty of applying the boundary conditions. However, comparison with the results for $\alpha = 0$ suggests an accuracy of about 1 %.

Finally the values of $p(\infty)$ and $[\eta q](\infty)$ determine the components of magnetic field on $y = 0$, since

$$\left.\begin{aligned} b_y(x, 0) &= p(\infty),\\ \text{and}\qquad b_x(|x|, 0) &= [\eta q](\infty).\end{aligned}\right\} \tag{B 13}$$

Appendix C

The solution given by (3.83), to the problem posed in § 3.3 is determined by solving the magnetic induction equation (the modified time dependent form of (1.35)),

$$\frac{\partial \chi}{\partial t} - \frac{\alpha}{r^2}\left(x\frac{\partial \chi}{\partial x} + y\frac{\partial \chi}{\partial y}\right) = \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2}, \tag{C 1}$$

subject to the boundary conditions,

$$\left.\begin{aligned} \chi &\to -x \quad \text{as} \quad r/t^{\frac{1}{2}} \to \infty, \\ \chi &= 0 \quad \text{at} \quad r = 0, \end{aligned}\right\} \tag{C 2}$$

where the time scale has been suitably non-dimensionalized, and

$$\mathbf{u} = (-x/r^2, -y/r^2).$$

Substituting

$$\chi = -xF(r^2/4t), \tag{C 3}$$

into (C 1) leads to the ordinary differential equation,

$$4\zeta^2 F'' + \{(8+2\alpha)\zeta + 4\zeta^2\}F' + \alpha F = 0, \tag{C 4}$$

where $\zeta = r^2/4t$. This equation has the solution

$$F(\zeta) = \frac{\Gamma\left(\dfrac{1-\frac{3}{2}m}{\frac{1}{2}-m}\right)}{\Gamma\left(\dfrac{(1-m)^2}{\frac{1}{2}-m}\right)}\zeta^{-m}\,\Phi\left(-m, \frac{(1-m)^2}{\frac{1}{2}-m}; -\zeta\right), \tag{C 5}$$

satisfying the boundary condition (C 2), where Φ is Humbert's symbol denoting a confluent hypergeometric function (Erdelyi *et al.* 1953), and

$$m = \tfrac{1}{4}\{(\alpha+2) - (\alpha^2+4)^{\frac{1}{2}}\}. \tag{C 6}$$

Thus, as $t \to \infty$ for fixed r,

$$\chi \sim (-1)\frac{\Gamma\left(\dfrac{1-\frac{3}{2}m}{\frac{1}{2}-m}\right)}{\Gamma\left(\dfrac{(1-m)^2}{\frac{1}{2}-m}\right)}\frac{4^m t^m x}{r^{2m}}. \tag{C 7}$$

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